Periodica Mathematica Hungarica Vol. 64 (1), 2012, pp. 101–117
DOI: 10.1007/s10998-012-9101-9

NUMBER OF SURVIVORS IN THE PRESENCE OF A DEMON

Guy Louchard[1], Helmut Prodinger[2] and Mark Daniel Ward[3]

[1]Département d'Informatique, Université Libre de Bruxelles
CP 212, Boulevard du Triomphe, B-1050, Bruxelles, Belgium
E-mail: louchard@ulb.ac.be

[2]Department of Mathematics, University of Stellenbosch
7602 Stellenbosch, South Africa
E-mail: hproding@sun.ac.za

[3]Department of Statistics, Purdue University
150 North University St., West Lafayette, IN 47907–2067, USA
E-mail: mdw@purdue.edu

(*Received May 3, 2010; Accepted June 10, 2010*)

[Communicated by Bálint Tóth]

Abstract

This paper complements the analysis of Louchard and Prodinger [LP08] on the number of *rounds* in a coin-flipping selection algorithm that occurs in the presence of a demon. We precisely analyze a very different aspect of the selection algorithm, using different methods of analysis. Specifically, we obtain precise descriptions of the distribution and all moments of the number of *participants* ultimately selected during the execution of the algorithm. The selection algorithm is robust in at least two significant ways. The presence of a demon allows for the precise analysis even when errors may occur between the rounds of the selection process. (The analysis also handles the more traditional case, in which no demon is involved.) The selection algorithm can also use either biased or unbiased coins.

1. Introduction

We precisely analyze the number of *survivors* in a selection process that occurs in the presence of a demon. Louchard and Prodinger [LP08] recently utilized a different methodology (for extreme value distributions, often referred to as "Gumbel

Mathematics subject classification numbers: 05A16, 05A30, 60C05, 68W20, 68W40.

Key words and phrases: analysis of algorithms, leader election, coin flip, survivor, demon, asymptotic, approximation, trie, q-Pochhammer symbol, factorial moment, distribution, generating function, recurrence, poissonization.

0031-5303/2012/$20.00

Akadémiai Kiadó, Budapest
Springer, Dordrecht

distributions") to analyze the number of *rounds* required to perform the selection algorithm.

The inclusion of a "demon" can be viewed as a generalization of traditional selection algorithms. The demon represents errors which might occur between rounds of the process. Another interpretation is that participants might be likely to drop out of the selection process for reasons unrelated to the coin flips in the selection process itself. In each round, exactly one participant is removed by the "demon" with probability ν; otherwise, the demon does not affect any participants in that round. Thus, a traditional selection algorithm (with no demon involved) is just a special case (using $\nu = 0$) of our very general analysis. The special case $\nu = 0$ (i.e., with no demon involved) is a selection process using a traversal of binary retrieval trees (tries), where a coin flip of "heads" is analogous to descending one direction in the trie, and a flip of "tails" corresponds to descending in the other direction. We use p and q (respectively) for the probabilities of heads and tails on coins in the selection algorithm. The analysis is sufficiently general to handle both the unbiased ($p = q = 1/2$) and biased ($p \neq q$) processes. The involvement of a demon makes the present algorithm more complicated and realistic than the traditional trie algorithm.

We are able to give the complete distribution and all moments of the number of survivors in a selection algorithm that occurs in the presence of a demon.

2. Selection algorithm

At the start of the selection algorithm, n people are present. Each person flips a coin with probability q of heads and p of tails. If all n people flip tails, then they are all selected by the algorithm and the selection process is finished. If $j > 0$ people flip heads, then these j people remain in play, and the other $n - j$ (who flipped tails) are eliminated from further play.

Then a demon arrives and, with probability ν, removes exactly one of the survivors, so $j-1$ remain; he leaves the j survivors alone with probability $\mu = 1-\nu$. If he leaves the j survivors alone, then these j survivors begin another round of coin flipping. If he removes a survivor and $j - 1 = 0$ (i.e., no survivors remain) then the selection process is finished and nobody is selected. If he removes a survivor and $j - 1 > 0$ (i.e., some survivors remain) then these $j - 1$ survivors begin another round of coin flipping.

The end of the algorithm can occur in two possible ways:

(1) During a round of coin flipping, one or more people remain. All of the remaining people simultaneously flip tails at this stage, and the algorithm ends. All of the people at this last stage are selected by the algorithm.

(2) During a visit by the demon, only one person is present, and this person is removed by the demon. In this case, zero people are selected by the algorithm.

All of the coin flips (among the people and among the rounds) are conducted independently.

EXAMPLE 2.1. Suppose that the probability of heads is $q = 1/3$ and the probability of tails is $p = 2/3$. Suppose that the demon appears with probability $\nu = 1/5$. Then the selection algorithm might proceed as follows: Initially 100 people are present. Exactly 31 of them flip heads (this happens with probability $\binom{100}{31}q^{31}p^{69}$), and then the demon arrives (this happens with probability 1/5) and removes one of the 31 survivors. So the next round begins with 30 people. Exactly 12 of the remaining 30 people flip heads (this happens with probability $\binom{30}{12}q^{12}p^{18}$), and then the demon leaves the survivors alone (this happens with probability 4/5). So the next round begins with 12 people. Exactly 3 of the remaining 12 people flip heads (this happens with probability $\binom{12}{3}q^{3}p^{9}$), and then the demon leaves the survivors alone (this happens with probability 4/5). So the next round begins with 3 people. Exactly 3 remaining people flip tails (this happens with probability $\binom{3}{0}q^{0}p^{3}$), and all three are selected by the algorithm.

3. Notation table

Most of the following definitions are already embedded at the appropriate places in the analysis. For the reader's convenience, we also summarize many of the terminologies used, in one succinct location. For the convenience of someone who already read [LP08], we preserve some of Louchard and Prodinger's earlier notation:

$n :=$ number of people present at the start of the selection algorithm,

$q :=$ probability that a coin flip shows heads,

$p := 1 - q$, probability that a coin flip shows tails,

$\nu :=$ probability that, during a visit by the demon, one survivor is removed,

$\mu := 1-\nu$, probability that the demon does not remove a survivor during a visit,

$X_n :=$ number of people selected by the algorithm, with n initial participants,

$\pi(n, m, j) :=$ probability the algorithm selects m of the initial n people and requires j rounds; by convention, $\pi(0, 0, 1) = 1$, and otherwise $\pi(0, m, j) = 0$,

$\pi(n, m) := \mathbf{P}(X_n = m) =$ probability that the algorithm selects m of the initial n people; by convention, $\pi(0, 0) = 1$, and $\pi(0, m) = 0$ for $m \neq 0$,

$$
\begin{aligned}
F_n(u,v) &:= \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\pi(n,m,j)u^m v^j,\\
F_n(u) &:= F_n(u,1) = \sum_{m=0}^{\infty}\pi(n,m)u^m = \sum_{m=0}^{\infty}\mathbf{P}(X_n = m)u^m,\\
Q &:= 1/q,\\
L &:= \ln Q,\\
\chi_\ell &:= 2\ell\pi\mathrm{i}/L,\\
H_j &:= \sum_{k=1}^{j}\frac{1}{k} \text{ is the } j\text{th harmonic number},\\
x^{\underline{j}} &:= \prod_{\ell=0}^{j-1}(x-\ell) = (x)(x-1)(x-2)\cdots(x-j+1) \text{ is the } j\text{th falling}\\
&\quad\ \text{power of } x,\\
\mathbb{E}[X_n^{\underline{j}}] &:= \mathbb{E}\Big[\prod_{\ell=0}^{j-1}(X_n-\ell)\Big] \text{ is the } j\text{th factorial moment of the random}\\
&\quad\ \text{variable } X_n,\\
F_n^{(s)}(u) &:= \frac{d^s}{du^s}F_n(u).
\end{aligned}
$$

We utilize some concepts from the theory of q-analysis. Since the value of q is fixed, we suppress the dependence on q. For positive integers n, we use the q-Pochhammer symbol

$$(x)_n := \prod_{j=0}^{n-1}(1-xq^j) = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}).$$

We also define $(x)_\infty := \prod_{j=0}^{\infty}(1-xq^j) = \lim_{n\to\infty}(x)_n$. For complex-valued z, we define

$$(x)_z := \frac{(x)_\infty}{(xq^z)_\infty}.$$

4. Results

The following two theorems precisely characterize the sth factorial moment $\mathbb{E}[X_n^{\underline{s}}]$ and the distribution $\mathbf{P}(X_n > r)$ of X_n. Each has the form const $+\,\delta(\log_Q n)+o(1)$. The constant and the function δ both depend on s or r, respectively. In both cases, the δ is fluctuating, because $e^{2\ell\pi\mathrm{i}\log_Q n}$ is fluctuating, with $|e^{2\ell\pi\mathrm{i}\log_Q n}| = 1$.

THEOREM 4.1. *The sth factorial moment of the number X_n of people selected by the algorithm, when beginning with n participants, is*

$$\begin{aligned}
&\mathbb{E}\,[X_n^{\underline{s}}]\\
&\quad= \frac{(Qp)^s}{L}\Big[\frac{(\mu q)_\infty}{(q)_\infty}(s-1)!+\\
&\qquad\qquad +s(-1)^{s-1}\sum_{j\geq s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\geq 1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}}-\sum_{\ell\geq 1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big)+\\
&\qquad\qquad\qquad +\Big(\frac{(\mu q)_\infty}{(q)_\infty}-\frac{(\mu q)_j}{(q)_j}\Big)(H_j-H_{j-s+1})\Big]\Big]+\\
&\qquad +\sum_{\ell\neq 0}\phi_{s,\ell}(n)+O(n^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
\phi_{s,\ell}(n) = \frac{(Qp)^s(-1)^s}{L}\Big(\frac{(\mu q)_\infty}{(q)_\infty}\chi_\ell^{\underline{s}}+s\sum_{j\geq 0}\Big(\frac{(\mu q)_j}{(q)_j}-\frac{(\mu q)_\infty}{(q)_\infty}\Big)\Big(j^{\underline{s-1}}-(j+\chi_\ell)^{\underline{s-1}}\Big)\Big)\times\\
\times\,\Gamma(-\chi_\ell)e^{2\ell\pi\mathrm{i}\log_Q n}.
\end{aligned}$$

THEOREM 4.2. *The distribution of the number X_n of people selected by the algorithm, when beginning with n participants, is*

$$\begin{aligned}
\mathbf{P}(X_n>r) = \frac{(\mu q)_\infty}{L(q)_\infty}\Big[L-\sum_{s=1}^{r}\frac{p^s}{s}-\frac{\nu p^r}{r+1}-Q(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times\\
\times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}}+\frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big]+\\
+\sum_{\ell\neq 0}\Phi_{r,\ell}(n)+O(n^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
&\Phi_{r,\ell}(n)\\
&\quad=\frac{Q}{L}\frac{(\mu q)_\infty}{(q)_\infty}\Big(-q\sum_{s=1}^{r}\frac{\chi_\ell^{\underline{s}}}{s!}(-p)^s+(-p)^r\nu q\frac{\chi_\ell^{\underline{r+1}}}{(r+1)!}+\\
&\qquad+(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\Big(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}}-\sum_{s=0}^{r}\frac{\chi_\ell^{\underline{s}}}{s!}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}}\Big)\Big)\times\\
&\quad\times\Gamma(-\chi_\ell)e^{2\ell\pi\mathrm{i}\log_Q n}.
\end{aligned}$$

5. Asymptotic moments of the number of survivors

5.1. Derivation of generating functions

We next establish an exact formula for the bivariate generating function $F_n(u,v)$ that describes the probabilities associated with the number of survivors and the number of rounds in the entire algorithm.

LEMMA 5.1. *Let $F_n(u,v)$ be a bivariate generating function such that the coefficient of $u^m v^j$ is the probability that, in the algorithm, exactly m people are ultimately selected and exactly j rounds are used to complete the election. Then*

$$F_n(u,v) = \sum_{k=0}^{n} \binom{n}{k} \Big(v(-q)^k \frac{(v)_k}{(\mu v q)_k} + (-q)^{k-1} \frac{(vq)_{k-1}}{(\mu q v)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j pv(u-1)(\mu q v)_j}{q^j (vq)_j} \Big). \tag{1}$$

The proof of Lemma 5.1 utilizes some recurrences associated with $F_n(u,v)$. The proof is given in Section 7.

COROLLARY 5.2. *Setting $u=1$ in Lemma 5.1, we obtain*

$$F_n(1,v) = \sum_{k=0}^{n} \binom{n}{k} v(-q)^k \frac{(v)_k}{(\mu v q)_k}. \tag{2}$$

This verifies that our results about the number of rounds agrees with the results from our previous paper.

During the remainder of the paper, we no longer pay attention to the number of rounds. We focus exclusively on the number of survivors.

LEMMA 5.3. *The sth factorial moment of X_n is*

$$\mathbb{E}[X_n^{\underline{s}}] = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \varphi_s(k), \tag{3}$$

with

$$\varphi_s(z) = q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s (-1)^{s-1} \psi_s(z), \tag{4}$$

and

$$\psi_s(z) = \frac{(\mu q)_\infty}{(q)_\infty} \frac{z^{\underline{s}}}{s} + \sum_{j\ge 0} \Big[\Big(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \Big) j^{\underline{s-1}} - \Big(\frac{(\mu q)_{j+z}}{(q)_{j+z}} - \frac{(\mu q)_\infty}{(q)_\infty} \Big) (j+z)^{\underline{s-1}} \Big].$$

5.2. Asymptotics

Now we turn our attention to the asymptotic moments of the number of survivors in the algorithm as the number n of initial participants grows large. We note that $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varphi_s(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2\ell\pi i}{L}$ for $\ell, m \in \mathbb{Z}$ with $m \le 0$. By [FS95, Theorem 2], we can restrict attention to the poles, where $m = 0$, i.e., where $z = \chi_\ell$ for $\ell \in \mathbb{Z}$. Thus

$$\mathbb{E}[X_n^{\underline{s}}] = \sum_{\ell\in\mathbb{Z}} \operatorname*{Res}_{z=\chi_\ell} \Big[\varphi_s(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\Big] + O(n^{-1}).$$

We need the local expansion of $\varphi_s(z)$ and thus $\psi_s(z)$ around $z = 0$ to two terms, since $\varphi_s(z)\frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z = 0$, but only a simple pole at $z = \chi_\ell$ for $\ell \ne 0$. As $z \to 0$,

$$\frac{(\mu q)_{j+z}}{(q)_{j+z}} \sim \frac{(\mu q)_j}{(q)_j}\Big[1 - zL\sum_{\ell\ge1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} + zL\sum_{\ell\ge1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big],$$

and

$$(j+z)^{\underline{s-1}} \sim j^{\underline{s-1}}\left[1 + z(H_j - H_{j-s+1})\right].$$

Thus

$$\begin{aligned}\psi_s(z) \sim{}& \frac{(\mu q)_\infty}{(q)_\infty}\frac{(-1)^{s-1}(s-1)!}{s}z+\\ &+\sum_{j\ge s-1}\Big[\Big(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big)j^{\underline{s-1}} -\\ &\qquad -\Big(\frac{(\mu q)_j}{(q)_j}\Big[1 - zL\sum_{\ell\ge1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} + zL\sum_{\ell\ge1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big] - \frac{(\mu q)_\infty}{(q)_\infty}\Big)\times\\ &\qquad\qquad \times j^{\underline{s-1}}[1 + z(H_j - H_{j-s+1})]\Big].\end{aligned}$$

More simply, as $z \to 0$,

$$\begin{aligned}\psi_s(z) \sim z\Big[&\frac{(\mu q)_\infty}{(q)_\infty}\frac{(-1)^{s-1}(s-1)!}{s}+\\ &+\sum_{j\ge s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\ge1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} - \sum_{\ell\ge1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big)+\\ &\qquad +\Big(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j}\Big)(H_j - H_{j-s+1})\Big]\Big].\end{aligned}$$

Notice the absence of the constant term! Substituting into the definition of $\varphi_s(z)$ in (4), it follows that

$$\varphi_s(z) \sim q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s(-1)^{s-1} \times \\ \times z\Big[\frac{(\mu q)_\infty}{(q)_\infty}\frac{(-1)^{s-1}(s-1)!}{s} + \\ + \sum_{j\ge s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\ge 1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} - \sum_{\ell\ge 1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big) + \\ + \Big(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j}\Big)(H_j - H_{j-s+1})\Big]\Big]$$

as $z \to 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\varphi_s(z) \sim \frac{(Qp)^s}{L}\Big[\frac{(\mu q)_\infty}{(q)_\infty}(s-1)! + \\ + s(-1)^{s-1}\sum_{j\ge s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\ge 1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} - \sum_{\ell\ge 1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big) + \\ + \Big(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j}\Big)(H_j - H_{j-s+1})\Big]\Big]$$

as $z \to 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \to 0$. Therefore

$$\operatorname*{Res}_{z=0}\Big[\varphi_s(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\Big] \\ = \lim_{z\to 0}\varphi_s(z) \\ = \frac{(Qp)^s}{L}\Big[\frac{(\mu q)_\infty}{(q)_\infty}(s-1)! + \\ + s(-1)^{s-1}\sum_{j\ge s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\ge 1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} - \sum_{\ell\ge 1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big) + \\ + \Big(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j}\Big)(H_j - H_{j-s+1})\Big]\Big].$$

So the sth factorial moment $\mathbb{E}[X_n^{\underline{s}}]$ of X_n is

$$\mathbb{E}[X_n^{\underline{s}}] = \frac{(Qp)^s}{L}\Big[\frac{(\mu q)_\infty}{(q)_\infty}(s-1)! + \\ + s(-1)^{s-1}\sum_{j\ge s-1} j^{\underline{s-1}}\Big[L\frac{(\mu q)_j}{(q)_j}\Big(\sum_{\ell\ge 1}\frac{\mu q^{j+\ell}}{1-\mu q^{j+\ell}} - \sum_{\ell\ge 1}\frac{q^{j+\ell}}{1-q^{j+\ell}}\Big) + \\ + \Big(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j}\Big)(H_j - H_{j-s+1})\Big]\Big] + \\ + \sum_{\ell\ne 0}\tilde{\phi}_{s,\ell}(n) + O(n^{-1}),$$

where

$$\begin{aligned}
\tilde{\phi}_{s,\ell}(n) &= \operatorname*{Res}_{z=\chi_\ell}\left[\varphi_s(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\right]\\
&= \operatorname*{Res}_{z=\chi_\ell}\,[(q)_{z-1}]\,\frac{q^{\chi_\ell}sp^s}{(\mu q)_{\chi_\ell}q^s}(-1)^{s-1}\psi_s(\chi_\ell)\frac{n!(-1)^n}{(\chi_\ell)(\chi_\ell-1)\cdots(\chi_\ell-n)}\\
&= \frac{(Qp)^s(-1)^s}{L}\Big(\frac{(\mu q)_\infty}{(q)_\infty}\chi_\ell^{\underline{s}}+s\sum_{j\geq 0}\Big(\frac{(\mu q)_j}{(q)_j}-\frac{(\mu q)_\infty}{(q)_\infty}\Big)\Big(j^{\underline{s-1}}-(j+\chi_\ell)^{\underline{s-1}}\Big)\Big)\times\\
&\quad\times\Gamma(-\chi_\ell)e^{2\ell\pi\mathrm{i}\log_Q n}\left(1+O(n^{-1})\right).
\end{aligned}$$

Note that $e^{2\ell\pi\mathrm{i}\log_Q n}$ is fluctuating, with $|e^{2\ell\pi\mathrm{i}\log_Q n}|=1$. This completes the proof of Theorem 4.1.

6. Asymptotic distribution of the number of survivors

6.1. Derivation of the distribution of the number of survivors

Now we derive an exact formula for the distribution of the number of survivors selected at the end of the algorithm.

LEMMA 6.1. *Let $r\geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is*

$$\begin{aligned}
\mathbf{P}(X_n>r)=&\sum_{k=1}^{n}\binom{n}{k}(-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}p^{r+1}(-1)^r\times\\
&\times\frac{(\mu q)_\infty}{(q)_\infty}\sum_{m\geq 0}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\sum_{j=0}^{k-1}\binom{j}{r}\left(q^j\right)^{m-1}.
\end{aligned}\tag{5}$$

The proof of Lemma 6.1 utilizes the q-binomial theorem; see Section 7.

LEMMA 6.2. *The distribution of X_n has the form*

$$\mathbf{P}(X_n>r)=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\varrho_r(k),\tag{6}$$

with

$$\varrho_r(z)=q^{z-1}\frac{(q)_{z-1}}{(\mu q)_z}p^{r+1}(-1)^r\Psi_r(z),\tag{7}$$

and

$$\begin{aligned}
\Psi_r(z)=&\frac{(\mu q)_\infty}{(q)_\infty}\Big(\frac{q}{(-p)^{r+1}}\Big(1-\sum_{s=0}^{r}\frac{z^{\underline{s}}}{s!}q^{-z}(-p)^s\Big)+\frac{(\mu-1)q}{p}\frac{z^{\underline{r+1}}}{(r+1)!}+\\
&+\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\Big(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}}-\sum_{s=0}^{r}\frac{z^{\underline{s}}}{s!}\frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}}\Big)\Big).
\end{aligned}$$

6.2. Asymptotics

Now we turn our attention to the asymptotic distribution of the number of survivors in the algorithm as the number n of initial participants grows large. We follow the derivation for the Rice Method discussed in Section 5. As before, $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varrho_r(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2\ell\pi i}{L}$ for $\ell, m \in \mathbb{Z}$ with $m \le 0$. Again, by [FS95], we focus on the poles $z = \chi_\ell$ for $\ell \in \mathbb{Z}$. Thus

$$\mathbf{P}(X_n > r) = \sum_{\ell\in\mathbb{Z}} \operatorname*{Res}_{z=\chi_\ell} \Big[\varrho_r(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\Big] + O(n^{-1}).$$

Similarly to the derivation in Section 5, we need the local expansion of $\varrho_r(z)$ and thus $\Psi_r(z)$ around $z=0$ to two terms, since $\varrho_r(z)\frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z=0$, but only a simple pole at $z=\chi_\ell$ for $\ell \neq 0$. As $z \to 0$,

$$\sum_{s=0}^{r} \frac{z^{\underline{s}}}{s!} q^{-z}(-p)^s \sim 1 - z\sum_{s=1}^{r}\frac{p^s}{s} + zL,$$

$$\frac{z^{\underline{r+1}}}{(r+1)!} \sim z\frac{(-1)^r}{(r+1)},$$

and

$$\sum_{s=0}^{r} \frac{z^{\underline{s}}}{s!}\frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}}$$
$$\sim \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} + z\sum_{s=1}^{r}\frac{(-1)^{s-1}}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} - z\frac{q^{(m-1)r}L(m-1)}{(1-q^{m-1})^{r+1}}.$$

Thus

$$\Psi_r(z) \sim \frac{(\mu q)_\infty}{(q)_\infty}\Big(\frac{q}{(-p)^{r+1}}\Big(z\sum_{s=1}^{r}\frac{p^s}{s} - zL\Big) + \frac{(\mu-1)q}{p}z\frac{(-1)^r}{(r+1)} +$$
$$+ \sum_{m\ge 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times$$
$$\times\Big(-z\sum_{s=1}^{r}\frac{(-1)^{s-1}}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + z\frac{q^{(m-1)r}L(m-1)}{(1-q^{m-1})^{r+1}}\Big)\Big).$$

More simply, as $z \to 0$,

$$\Psi_r(z) \sim z\frac{(\mu q)_\infty}{(q)_\infty}\Big[\frac{q(-1)^r}{p^{r+1}}\Big(L - \sum_{s=1}^{r}\frac{p^s}{s}\Big) - \frac{(1-\mu)q}{p}\frac{(-1)^r}{(r+1)} +$$
$$+ \sum_{m\ge 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times$$
$$\times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big].$$

As before, notice the absence of the constant term. Substitution into (7) yields

$$\begin{aligned}\varrho_r(z) \sim q^{z-1}\frac{(q)_{z-1}}{(\mu q)_z}p^{r+1}(-1)^r \times \\ \times z\frac{(\mu q)_\infty}{(q)_\infty}\Big[\frac{q(-1)^r}{p^{r+1}}\Big(L-\sum_{s=1}^{r}\frac{p^s}{s}\Big) - \frac{(1-\mu)q}{p}\frac{(-1)^r}{(r+1)} + \\ + \sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times \\ \times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big]\end{aligned}$$

as $z \to 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\begin{aligned}\varrho_r(z) \sim \frac{(\mu q)_\infty}{L(q)_\infty}\Big[L-\sum_{s=1}^{r}\frac{p^s}{s} - \frac{\nu p^r}{r+1} - \\ -Q(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times \\ \times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big]\end{aligned}$$

as $z \to 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \to 0$. Therefore

$$\begin{aligned}&\operatorname*{Res}_{z=0}\Big[\varrho_r(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\Big] \\ &\quad = \lim_{z\to 0}\varrho_r(z) \\ &\quad = \frac{(\mu q)_\infty}{L(q)_\infty}\Big[L-\sum_{s=1}^{r}\frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times \\ &\qquad \times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big].\end{aligned}$$

So

$$\begin{aligned}\mathbf{P}(X_n > r) = \frac{(\mu q)_\infty}{L(q)_\infty}\Big[L-\sum_{s=1}^{r}\frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\times \\ \times\Big(\sum_{s=1}^{r}\frac{(-1)^s}{s}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}}\Big)\Big] + \\ + \sum_{\ell\neq 0}\tilde{\Phi}_{r,\ell}(n) + O(n^{-1}),\end{aligned}$$

where

$$\begin{aligned}
&\tilde{\Phi}_{r,\ell}(n)\\
&\quad= \operatorname*{Res}_{z=\chi_\ell}\left[\varrho_r(z)\frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)}\right]\\
&\quad= \operatorname*{Res}_{z=\chi_\ell}\left[(q)_{z-1}\right]\frac{q^{\chi_\ell-1}p^{r+1}}{(\mu q)_{\chi_\ell}}(-1)^r\Psi_r(\chi_\ell)\frac{n!(-1)^n}{(\chi_\ell)(\chi_\ell-1)\cdots(\chi_\ell-n)}\\
&\quad= \frac{Q}{L}\frac{(\mu q)_\infty}{(q)_\infty}\Big(-q\sum_{s=1}^{r}\frac{\chi_\ell^s}{s!}(-p)^s+(-p)^r\nu q\frac{\chi_\ell^{r+1}}{(r+1)!}+\\
&\qquad+(-p)^{r+1}\sum_{m\geq 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\Big(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}}-\sum_{s=0}^{r}\frac{\chi_\ell^s}{s!}\frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}}\Big)\Big)\times\\
&\qquad\times\Gamma(-\chi_\ell)e^{2\ell\pi\mathrm{i}\log_Q n}\left(1+O(n^{-1})\right).
\end{aligned}$$

Note that $e^{2\ell\pi\mathrm{i}\log_Q n}$ is fluctuating, with $|e^{2\ell\pi\mathrm{i}\log_Q n}|=1$. This completes the proof of Theorem 4.2

7. Proofs

Proof of Lemma 5.1. When starting with n participants, if all n participants are simultaneously eliminated by coin flipping, then these n participants are selected by the algorithm; this corresponds to the term $\binom{n}{0}q^0p^nu^nv$ in recurrence (8) below. If exactly j participants obtain heads, with $1\leq j\leq n$, then the demon arrives and removes one additional participant with probability ν, or leaves the j remaining participants alone with probability μ. This phenomenon corresponds to $v\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}(\nu F_{j-1}(u,v)+\mu F_j(u,v))$ in formula (8). (Note that, since $F_0(u,v)=v$, the recurrence below also holds when $n=0$.) So the recurrence

$$F_n(u,v)=\binom{n}{0}q^0p^nu^nv+v\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}(\nu F_{j-1}(u,v)+\mu F_j(u,v)) \tag{8}$$

holds for all integers $n\geq 0$. More simply,

$$F_n(u,v)=v\bigg(p^nu^n+\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}(\nu F_{j-1}(u,v)+\mu F_j(u,v))\bigg). \tag{9}$$

Next we define the exponential generating function

$$G(z,u,v):=\sum_{n=0}^{\infty}F_n(u,v)\frac{z^n}{n!}.$$

From the recurrence in (9), it follows that

$$\begin{aligned} G(z,u,v) &= v\sum_{n=0}^{\infty}\Big(p^n u^n + \sum_{j=1}^{n}\binom{n}{j} q^j p^{n-j}(\nu F_{j-1}(u,v)+\mu F_j(u,v))\Big)\frac{z^n}{n!} \\ &= v\Big(e^{puz} + \sum_{j=1}^{\infty} q^j z^j(\nu F_{j-1}(u,v)+\mu F_j(u,v))\sum_{n=j}^{\infty}\binom{n}{j}p^{n-j}\frac{z^{n-j}}{n!}\Big) \\ &= v\Big(e^{puz} + e^{pz}\sum_{j=1}^{\infty}\frac{(qz)^j}{j!}(\nu F_{j-1}(u,v)+\mu F_j(u,v))\Big) \\ &= v\Big(e^{puz} + e^{pz}\Big(\nu q\int G(qz,u,v)\,dz + \mu G(qz,u,v) - \mu v\Big)\Big). \end{aligned} \tag{10}$$

The generating function $G(z,u,v)$ becomes simpler if we replace the fixed number n of people present at the start of the algorithm by a Poisson number of participants with mean z. For this reason, we replace $G(z,u,v)$ by the Poissonized exponential generating function

$$D(z,u,v) := G(z,u,v)e^{-z} = \sum_{n=0}^{\infty} D_n(u,v)\frac{z^n}{n!}.$$

From (10), it follows that

$$D(z,u,v) = ve^{(pu-1)z} + ve^{-qz}\Big(\nu q\int G(qz,u,v)\,dz + \mu G(qz,u,v) - \mu v\Big). \tag{11}$$

We use a succinct notation for differentiation with respect to the first of three variables:

$$D'(z,u,v) := \frac{d}{dz}D(z,u,v)$$

and

$$G'(z,u,v) := \frac{d}{dz}G(z,u,v).$$

Differentiating both sides of (11) with respect to z yields

$$\begin{aligned} D'(z,u,v) &= (pu-1)ve^{(pu-1)z} - vqe^{-qz}\Big(\nu q\int G(qz,u,v)\,dz + \mu G(qz,u,v) - \mu v\Big) + \\ &\quad + ve^{-qz}\left(\nu q G(qz,u,v) + \mu q G'(qz,u,v)\right). \end{aligned}$$

It follows that

$$D'(z,u,v) + qD(z,u,v) = \mu qvD'(qz,u,v) + qvD(qz,u,v) + e^{(pu-1)z}pv(u-1).$$

For $n \geq 1$, extracting the coefficient of $\frac{z^{n-1}}{(n-1)!}$ from $D(z,u,v) = \sum_{n=0}^{\infty} D_n(u,v)\frac{z^n}{n!}$ yields

$$D_n(u,v) + qD_{n-1}(u,v) = \mu qvD_n(u,v)q^{n-1} + qvD_{n-1}(u,v)q^{n-1} + (pu-1)^{n-1}pv(u-1),$$

or equivalently,

$$D_n(u,v) = D_{n-1}(u,v)\frac{vq^n - q}{1-\mu vq^n} + \frac{(pu-1)^{n-1}pv(u-1)}{1-\mu vq^n}.$$

Iterating this recurrence yields

$$\begin{aligned}D_n(u,v) &= v(-q)^n\frac{(v)_n}{(\mu vq)_n} + \sum_{j=0}^{n-1}\frac{(pu-1)^j pv(u-1)\prod_{k=j+2}^n(vq^k-q)}{\prod_{\ell=j+1}^n(1-\mu vq^\ell)}\\ &= v(-q)^n\frac{(v)_n}{(\mu vq)_n} + (-q)^{n-1}\frac{(vq)_{n-1}}{(\mu qv)_n}\sum_{j=0}^{n-1}\frac{(1-pu)^j pv(u-1)(\mu qv)_j}{q^j(vq)_j}.\end{aligned}$$

Note that $F_n(u,v) = \sum_{k=0}^n \binom{n}{k} D_k(u,v)$, so Lemma 5.1 follows. □

Proof of Lemma 5.3. Setting $v=1$ in Lemma 5.1, it follows that

$$\begin{aligned}F_n(u) &= F_n(u,1)\\ &= \sum_{k=0}^n\binom{n}{k}\Big((-q)^k\frac{(1)_k}{(\mu q)_k} + (-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}\sum_{j=0}^{k-1}\frac{(1-pu)^j p(u-1)(\mu q)_j}{q^j(q)_j}\Big)\\ &= 1 + \sum_{k=1}^n\binom{n}{k}(-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}\sum_{j=0}^{k-1}\frac{(1-pu)^j p(u-1)(\mu q)_j}{q^j(q)_j}.\end{aligned}\tag{12}$$

It follows that, for $s \geq 1$,

$$F_n^{(s)}(1) = \sum_{k=1}^n\binom{n}{k}(-1)^{k-1}q^k\frac{(q)_{k-1}}{(\mu q)_k}s(Qp)^s(-1)^{s-1}\sum_{j=0}^{k-1}\frac{(\mu q)_j}{(q)_j}j^{\underline{s-1}}.\tag{13}$$

Dissecting the summation over j in (13), we obtain

$$\begin{aligned}&\sum_{j=0}^{k-1}\frac{(\mu q)_j}{(q)_j}j^{\underline{s-1}}\\ &\quad= \frac{(\mu q)_\infty}{(q)_\infty}\sum_{j=0}^{k-1}j^{\underline{s-1}} + \sum_{j=0}^{k-1}\Big[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big]j^{\underline{s-1}}\\ &\quad= \frac{(\mu q)_\infty}{(q)_\infty}\frac{k^{\underline{s}}}{s} + \sum_{j\geq 0}\Big[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big]j^{\underline{s-1}} - \sum_{j\geq k}\Big[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big]j^{\underline{s-1}}\\ &\quad= \frac{(\mu q)_\infty}{(q)_\infty}\frac{k^{\underline{s}}}{s} + \sum_{j\geq 0}\Big[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big]j^{\underline{s-1}} - \sum_{j\geq 0}\Big[\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty}\Big](j+k)^{\underline{s-1}}\\ &\quad= \frac{(\mu q)_\infty}{(q)_\infty}\frac{k^{\underline{s}}}{s} + \sum_{j\geq 0}\Big[\Big(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty}\Big)j^{\underline{s-1}} - \Big(\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty}\Big)(j+k)^{\underline{s-1}}\Big].\end{aligned}$$

Finally, we observe that, since $F_n(u) = \sum_{m=0}^{\infty} \pi(n,m)u^m$, then $F_n^{(s)}(1) = \sum_{m=0}^{\infty} m^{\underline{s}} \pi(n,m) = \mathbb{E}[X_n^{\underline{s}}]$. Thus $\mathbb{E}[X_n^{\underline{s}}]$ has the representation given in the statement of Lemma 5.3.

□

Proof of Lemma 6.1. First of all,

$$\mathbf{P}(X_n > r) = \sum_{m>r} \pi(n,m) = 1 - \sum_{m=0}^{r} \pi(n,m).$$

Note that $\sum_{m=0}^{r} \pi(n,m) = [u^r]\frac{F_n(u)}{1-u}$, and of course $1 = [u^r]\frac{1}{1-u}$, so

$$\begin{aligned}
\mathbf{P}(X_n > r) &= [u^r]\frac{1-F_n(u)}{1-u} \\
&= [u^r]\frac{F_n(u)-1}{u-1} \\
&= [u^r]\sum_{k=1}^{n}\binom{n}{k}(-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}\sum_{j=0}^{k-1}\frac{(1-pu)^j p(\mu q)_j}{q^j(q)_j} \quad \text{(by equation (12))} \\
&= \sum_{k=1}^{n}\binom{n}{k}(-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}p^{r+1}(-1)^r\sum_{j=0}^{k-1}\binom{j}{r}\frac{(\mu q)_j}{q^j(q)_j}.
\end{aligned} \tag{14}$$

We focus on the second summation in (14). Recall that $(x)_z := (x)_\infty/(xq^z)_\infty$, so

$$\frac{(\mu q)_j}{(q)_j} = \frac{(\mu q)_\infty}{(q)_\infty}\frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty}. \tag{15}$$

Also, the q-binomial theorem states $\frac{(az)_\infty}{(z)_\infty} = \sum_{m\geq 0}\frac{(a)_m}{(q)_m}z^m$. Specifying $z = \mu q^{j+1}$ and $a = 1/\mu$,

$$\frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty} = \sum_{m\geq 0}\frac{(1/\mu)_m}{(q)_m}\left(\mu q^{j+1}\right)^m. \tag{16}$$

Combining (15) and (16) yields

$$\begin{aligned}
\sum_{j=0}^{k-1}\binom{j}{r}\frac{(\mu q)_j}{q^j(q)_j} &= \sum_{j=0}^{k-1}\binom{j}{r}q^{-j}\frac{(\mu q)_\infty}{(q)_\infty}\sum_{m\geq 0}\frac{(1/\mu)_m}{(q)_m}\left(\mu q^{j+1}\right)^m \\
&= \frac{(\mu q)_\infty}{(q)_\infty}\sum_{m\geq 0}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\sum_{j=0}^{k-1}\binom{j}{r}\left(q^j\right)^{m-1}.
\end{aligned} \tag{17}$$

Thus

$$\begin{aligned}
&\mathbf{P}(X_n > r) \\
&= \sum_{k=1}^{n}\binom{n}{k}(-q)^{k-1}\frac{(q)_{k-1}}{(\mu q)_k}p^{r+1}(-1)^r\frac{(\mu q)_\infty}{(q)_\infty}\sum_{m\geq 0}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\sum_{j=0}^{k-1}\binom{j}{r}\left(q^j\right)^{m-1},
\end{aligned}$$

as claimed in the lemma.

□

Proof of Lemma 6.2. Let $r \geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is

$$\mathbf{P}(X_n > r) = \sum_{k=1}^{n} \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m\geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} \left(q^j\right)^{m-1}. \tag{18}$$

We handle the sum over m in Lemma 6.1 in three parts, $m = 0$, $m = 1$, and $m \geq 2$, as follows:

$$\begin{aligned}\sum_{m\geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} \left(q^j\right)^{m-1} &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(1-1/\mu)(\mu q)}{(1-q)} \sum_{j=0}^{k-1} \binom{j}{r} + \sum_{m\geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} \left(q^j\right)^{m-1} \\ &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(\mu-1)q}{p} \frac{k^{\underline{r+1}}}{(r+1)!} + \sum_{m\geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} \left(q^j\right)^{m-1}.\end{aligned} \tag{19}$$

Now we focus our attention on the sums of the form $\sum_{j=0}^{k-1} \binom{j}{r} x^j$. Writing $D = \frac{d}{dx}$, we note

$$\sum_{j=0}^{k-1} \binom{j}{r} x^j = \frac{x^r}{r!} \sum_{j=0}^{k-1} j^{\underline{r}} x^{j-r} = \frac{x^r}{r!} \sum_{j=0}^{k-1} D^r x^j = \frac{x^r}{r!} D^r \sum_{j=0}^{k-1} x^j = \frac{x^r}{r!} D^r \frac{1-x^k}{1-x}. \tag{20}$$

The remainder of the analysis does not depend on k being an integer. We have

$$\begin{aligned}\frac{x^r}{r!} D^r \frac{1-x^k}{1-x} &= \frac{x^r}{r!} \sum_{s=0}^{r} \binom{r}{s} D^s (1-x^k) \cdot D^{r-s} \Big(\frac{1}{1-x}\Big) \\ &= \frac{x^r}{r!} (1-x^k) \cdot \frac{r!}{(1-x)^{r+1}} - \frac{x^r}{r!} \sum_{s=1}^{r} \binom{r}{s} k^{\underline{s}} x^{k-s} \cdot \frac{(r-s)!}{(1-x)^{r-s+1}} \\ &= \frac{x^r}{(1-x)^{r+1}} - \frac{x^r x^k}{(1-x)^{r+1}} - \sum_{s=1}^{r} \frac{k^{\underline{s}}}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}} \\ &= \frac{x^r}{(1-x)^{r+1}} - \sum_{s=0}^{r} \frac{k^{\underline{s}}}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}}.\end{aligned} \tag{21}$$

Thus, combining (20) and (21) with $x = q^{m-1}$, we can simplify the "$m \geq 2$" term of (19) as follows:

$$\sum_{j=0}^{k-1} \binom{j}{r} \left(q^j\right)^{m-1} = \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^{r} \frac{k^{\underline{s}}}{s!} \frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}. \tag{22}$$

For $m = 0$, the analogous equation is

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{j}{r}q^{-1} &= \frac{q^{-r}}{(1-q^{-1})^{r+1}} - \sum_{s=0}^{r}\frac{k^{\underline{s}}}{s!}\frac{q^{-(k+r-s)}}{(1-q^{-1})^{r-s+1}}\\ &= \frac{q}{(-p)^{r+1}}\Big(1-\sum_{s=0}^{r}\frac{k^{\underline{s}}}{s!}q^{-k}(-p)^s\Big).\end{aligned} \tag{23}$$

Plugging the results from (22) and (23) into (19), we get

$$\begin{aligned}&\sum_{m\ge 0}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\sum_{j=0}^{k-1}\binom{j}{r}\Big(q^j\Big)^{m-1}\\ &\quad= \frac{q}{(-p)^{r+1}}\Big(1-\sum_{s=0}^{r}\frac{k^{\underline{s}}}{s!}q^{-k}(-p)^s\Big) + \frac{(\mu-1)q}{p}\frac{k^{\underline{r+1}}}{(r+1)!} +\\ &\qquad + \sum_{m\ge 2}\frac{(1/\mu)_m(\mu q)^m}{(q)_m}\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^{r}\frac{k^{\underline{s}}}{s!}\frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}.\end{aligned}$$

Finally, a substitution into the form of $\mathbf{P}(X_n > r)$ in Lemma 6.1 yields Lemma 6.2.

□

8. Future problems

A key problem for future analysis involves a more robust demon, who might be able to remove more than one participant at a time. Another problem to be studied in the future might involve replacing the 2-outcome coins (heads versus tails) with a coin that itself involves some uncertainty. Another interpretation of this extension is that the parameters p and q are unknown before the coin is flipped. Many other possibilities exist for generalizing the present algorithm.

References

[FS95] P. Flajolet and R. Sedgewick, Mellin transforms and asymptotics: Finite differences and Rice's integrals, *Theoret. Comput. Sci.*, **144** (1995), 101–124.

[FS09] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge, 2009.

[Loè78] M. Loève, *Probability Theory* (2 volumes), 4th edition, Springer, New York, 1977, 1978.

[LP08] G. Louchard and H. Prodinger, Advancing in the presence of a demon, *Math. Slovaca*, **58** (2008), 263–276.

[Szp01] W. Szpankowski, *Average Case Analysis of Algorithms on Sequences*, Wiley, New York, 2001.